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# THE TENSILE STRENGTH OF A FIBER-REINFORCED CERAMIC

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**Abstract**—A means of theoretically predicting the ultimate tensile strength of a unidirectional, fiber-reinforced ceramic which undergoes multiple matrix cracking is presented. The analysis, which represents an elaboration of a theory recently presented by the authors, accounts for the presence of the matrix cracks, as well as for the random failure of individual fibers that occur with increasing likelihood as the applied stress is increased. The key material parameters are the fiber strength and strength variability, and the interfacial shear strength. In addition, the prediction can account for stress concentrations at points along the fiber surface where the matrix cracks impinge. Comparisons with experimental data and sensitivity analyses show that it is important to account for stress concentrations and to have an accurate value for the *in situ* fiber strength.

## INTRODUCTION

Ceramic–matrix composites appear to be gaining increased attention in the search for high strength, high temperature materials. In addition, many of the possible applications involve subjecting the material to hostile and aggressive environments; hence, there must be confidence that strength can be maintained under a variety of conditions. Of great concern currently is the influence of the environment on the fiber–matrix interface, which is often cited as a key determinant of composite strength.

This is a challenging problem, however, as the dependence of the composite strength on the interface properties is quite complex. In some systems, for example reaction bonded silicon nitride reinforced by silicon carbide monofilaments, oxidation which *reduces* the interfacial shear strength causes the composite strength to diminish (Bhatt, 1989). In other systems, for example lithium alumino-silicate reinforced by silicon carbide fibers, oxidation which *increases* the interfacial shear strength also causes the composite strength to diminish (Brennan, 1988). Finally, a silicon carbide fiber-reinforced silicon carbide exhibits first increasing and then decreasing tensile strength as a function of interfacial shear strength (Lowden, 1990). Clearly, these complicated observations will require a more comprehensive theory of composite strength than exists to date. One such theory is offered in the present paper, which elaborates substantially upon a model of ultimate tensile strength that was recently presented by the authors (Schwietert and Steif, 1990a).

## BACKGROUND

In order to appreciate the theory elaborated upon here, it is useful to consider previously proposed means of computing the ultimate strength, $\sigma_{UTS}$, of a fiber composite. Perhaps the simplest estimate of a composite's strength is the rule of mixtures. For a brittle-matrix composite in which the matrix fails first, the rule of mixtures would give the strength as

$$\sigma_{UTS} = V_f \sigma_f \tag{1}$$

where $V_f$ is the fiber volume fraction and $\sigma_f$ is the mean strength of fibers. Equation (1) would be an accurate estimate of the strength if the fibers all had identical strengths, and if the composite failure coincided with all fibers simultaneously breaking on a single matrix-crack plane.

Unfortunately, the assumptions underlying the rule of mixtures estimate are at odds with reality. First, real fibers exhibit a statistical variation in strength, which causes fiber breaks to be dispersed throughout the composite, as evidenced by the variety of pull-out lengths: failure generally does not occur on a single crack plane. Secondly, the statistical variation in strength depends on the fiber length. If $\sigma_f$ were taken to be the mean strength of fibers equal in length to the gauge length, then (1) would imply that the composite strength varies with gauge length to the same degree as the mean fiber strength varies with fiber length.

Consider, for example, the Weibull strength distribution as commonly applied to fibers. According to this distribution, the probability $P(\sigma)\,\mathrm{d}\sigma$ that a fiber of length $L$ has a strength between $\sigma$ and $\sigma+\mathrm{d}\sigma$ is given by

$$P(\sigma)\,\mathrm{d}\sigma = L\alpha m\sigma^{m-1}\exp[-L\alpha\sigma^m]\,\mathrm{d}\sigma$$

where $\alpha$ and $m$ are the standard Weibull parameters, the latter being related to the strength variability. (This probability can be derived from a flaw function which is introduced below.) The corresponding mean fiber strength is then given by

$$\sigma_f = \frac{\Gamma(1+1/m)}{(\alpha L)^{1/m}} \tag{2}$$

where $\Gamma(x)$ is the complete Gamma function with $\Gamma(1) = 1$. For fibers commonly in use, the Weibull modulus $m$ is often in the range of $3 < m < 9$, in which case there is a significant variation of fiber strength with length. However, there seems to be no experimental evidence that the *composite* strength is significantly size-dependent, in general.

A second means of estimating composite strength is to consider the composite as simply a bundle of fibers. In the classic bundle calculation, each fiber in the bundle is assumed to have the identical statistical distribution in strength. Furthermore, it is assumed that the load given up by a broken fiber in a bundle is taken up *equally* by the remaining intact fibers. For a Weibull distribution of fiber strengths, the asymptotic mean strength of a fiber bundle as the number of fibers in the bundle tends to infinity, $\sigma_B$, is given by (Daniels, 1945)

$$\sigma_B = \frac{1}{(\alpha meL)^{1/m}} \tag{3}$$

which is always less than the mean strength $\sigma_f$. If a composite with fiber volume fraction $V_f$ is treated as a fiber bundle, then its strength (assuming a large number of fibers) would be

$$\sigma_{UTS} = V_f\sigma_B. \tag{4}$$

One particular assumption in the bundle model practically prohibits its application to estimating the composite strength. Once a fiber in a bundle breaks, that fiber is assumed to carry no load whatsoever; by contrast, a broken fiber in a composite regains its load with distance from the break. This situation was salvaged by Rosen's (1965) ingenious chain of bundles model. This model involves, first, the notion of a fiber's ineffective length. Because the interfacial shear stress must act over some *length* in order to transfer the load to the fibers from the matrix, a portion of each fiber (near the fiber ends) is *ineffective*. More pertinent to composite strength is the ineffective portion of the fiber immediately adjacent to a fiber break. In reality, the load is gradually transferred from the matrix back to the fiber as a function of distance from the fiber break. Yet, it is useful to define an ineffective length as that portion of the fiber which carries less than, say, 90% of the full fiber load.

Rosen suggested that, insofar as strength is concerned, a composite is like a series or a chain of bundles, each bundle having length equal to the ineffective length (a schematic

Fig. 1a. Schematic of a composite with fiber breaks.

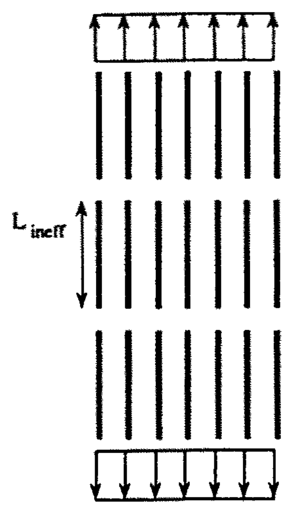


Fig. 1b. Schematic of Rosen's chain of bundles.

of this model is given in Fig. 1). From this suggestion, one can readily compute the composite strength to be

$$\sigma_{UTS} = \frac{V_f}{(\alpha m e L_{\mathrm{ineff}})^{1/m}} \tag{5}$$

where $L_{\mathrm{ineff}}$ is the ineffective length. (This estimate assumes an infinite number of fibers, which implies no statistical variability in the bundle strength; since each bundle in the chain has identical strength, the chain is precisely as strong as each bundle.) Clearly, this chain of bundles is not precisely the same as a composite in which fiber breaks appear scattered throughout the material. Nevertheless, the predictions of Rosen's model do agree with our intuition regarding one important feature: the more quickly the load can be transferred back to the fibers near a break (the shorter the ineffective length), the less detrimental will be the effect of the break on the composite's strength and, therefore, the higher will be the composite strength.

Two significant lines of research arose from Rosen's model, both of which implicitly question the validity of the assumptions which lead to eqn (5). First, there has been extensive consideration of the consequences of the fact that the number of fibers in a composite is actually finite (Phoenix and Taylor, 1973; Smith, 1982; McCartney and Smith, 1983). When a fiber bundle has a *finite* number of fibers, it has a distribution in strength, with complicated consequences for the chain of bundles. In a second line of research, the assumption of equal load sharing, which is implicit in the calculation of bundle strength, has been abandoned (see, for example, Zweben and Rosen, 1970; Argon, 1972). The opposite of equal load sharing is "local load sharing", in which the load "given up" by a breaking fiber is taken up mostly or exclusively by its neighbors, implying that fiber breaks might eventually tend to cluster.

But how valid are the assumptions of Rosen's basic chain-of-bundles model, in particular the issue of equal versus local load sharing? Consideration of typical fracture surfaces of ceramic–matrix composites under tensile loading suggests an answer to this question. Of interest, in particular, are tough, ceramic–matrix composites which exhibit multiple matrix cracking; with increased loading these composites eventually reach their ultimate strength followed by extensive pull-out. Generally, the fracture of these materials appears to coincide with continued separation of the composite across a single matrix–crack plane. Furthermore, a perusal of the pulled out fibers protruding from each half of the failed specimen often reveals that the pull-out lengths of different fibers are statistically independent of one another (compare the schematics in Figs 2a and b). If there were to be a correlation between the lengths of nearby fibers (as in Fig. 2a), then one would strongly suspect some degree of local load sharing: a break in one fiber would tend to produce a break roughly in the same location of a neighboring fiber.

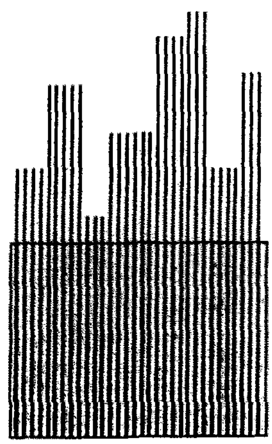

Fig. 2a. Schematic of pulled out fibers under conditions of local load sharing.

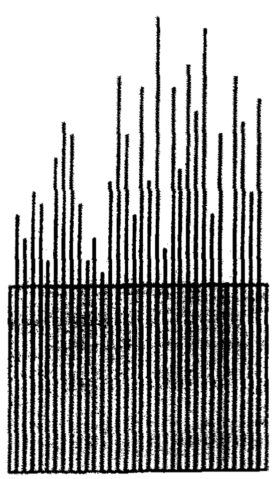

Fig. 2b. Schematic of pulled out fibers under conditions of equal load sharing.

Therefore, it seems reasonable to retain the assumption of equal load sharing, at least when applying a strength theory to composites in which the fiber pull-out lengths have the appearance of being statistically independent. In addition, we take the view, which will be borne out below, that much can be learned of the dependence of composite strength on significant parameters (on interface, for example) without introducing the additional complexities associated with the finiteness of the number of fibers. Hence, we are returning to the original assumptions of Rosen's simple chain-of-bundles model. Within these assumptions, however, we show that the ultimate tensile strength can be estimated in a way that remains faithful to the actual distribution of fiber breaks (as they appear schematically in Fig. 1a), without resorting to Rosen's reconfiguration of the composite into a chain of bundles. Our theory for tensile strength, which is an elaboration of one proposed recently by the authors (Schwietert and Steif, 1990a), is quite general, at least within the assumptions thus far discussed; as will be seen, it is capable of explaining a significant portion of the complicated dependence of strength on interface. A related approach has also been taken by Sutcu (1989), who tacitly begins at the same point as our eqn (6); the difference between the two approaches is indicated below.

## THEORY

A unidirectionally reinforced, fiber composite, which possesses a large (essentially infinite) number of long fibers and is subjected to uniaxial tension parallel to the fibers, is contemplated (see Fig. 3). It is assumed that at some level of loading matrix cracks begin to form normal to the fiber direction, and, with increasing load, the matrix cracks eventually saturate. At saturation, the matrix cracks are all perfectly flat, equally spaced and spanned by intact fibers. Thereafter, the only damage that occurs is in the form of fiber breaks (interface debonding can occur and is handled implicitly, as will be seen below). All fibers have the same statistical distribution in strength, and their failure is governed by the commonly adopted weakest link law. Hence, fiber breaks will appear dispersed throughout the composite. As fiber breaks accumulate, they continually diminish the load carrying capacity of the composite. Such a composite, having many fibers which are sufficiently long, has a definite ultimate strength which is independent of the specimen length. Our goal is to compute the ultimate strength of the composite.

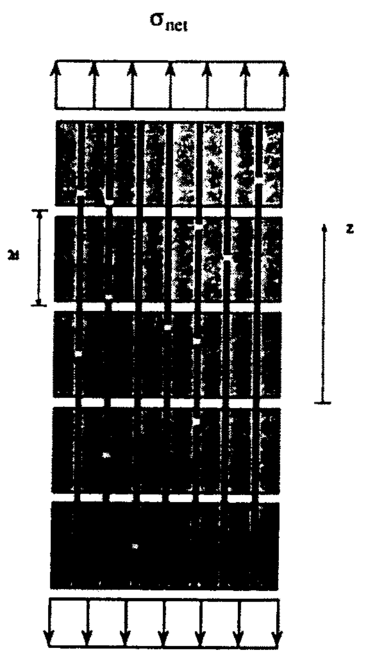


Fig. 3. Schematic of a composite under an average axial stress $\sigma_{net}$ which has sustained multiple matrix cracking.

The present theory envisions a standard tension test in which the grips which hold the specimen are separated at, say, a constant rate. The average stress in the specimen increases from zero up to a maximum stress and then decreases. The ultimate strength of the composite is found by computing the cross-sectionally averaged tensile stress in the composite at each *instant* of the tension test. The maximum value of this average tensile stress is our prediction of the ultimate strength. However, such a procedure requires a parameter which increases monotonically as the test proceeds (like the average strain).

A suitable parameter is available if we are prepared to generalize the notion of "equal load sharing" as follows. (In the ensuing discussion, the following convention is adopted: unless otherwise indicated, the stress at a point in a fiber means the longitudinal stress averaged over the fiber cross-section at the point.) It is assumed that, at a given remote strain, the stress distribution along a fiber is dependent only on the positions of the matrix cracks that it spans and on the longitudinal positions of *its* breaks; however, it is *not* dependent on the positions of breaks in *neighboring* fibers. This implies that all *intact* fibers have the same distribution of stress along their lengths. Since the stress in a fiber increases monotonically in a tensile test as long as it remains intact, we can parameterize the tensile test by the stress at some point in an intact fiber. Specifically, the stress in an intact fiber at a matrix–crack plane, which is denoted by $\sigma_I$, is chosen to parameterize the test. As shown below, all quantities necessary for computing the average tensile stress in the specimen can be found as functions of $\sigma_I$.

Since the average tensile stress in the specimen is the same at all planes, we can evaluate that stress on any plane that is convenient, in particular on one of the matrix crack planes (which are perfectly flat and equally spaced). This average tensile stress, $\sigma_{net}$, which is the average of the stresses carried by all the fibers across the matrix crack, can be written as

$$\sigma_{net} = V_f \left\{ \frac{1}{N_T} \sum_{j=1}^{N_I} (\sigma_I)_j + \frac{1}{N_T} \sum_{j=1}^{N_B} (\sigma_B)_j \right\} \tag{6}$$

where $N_T$ is the total number of fibers in the specimen, $N_I$ is the number of fibers in the specimen that are intact, and $N_B$ is the number of fibers in the specimen that are broken. $(\sigma_I)_j$ represents the tensile stress carried by the $j$th intact fiber at the plane of a matrix crack; consistent with the generalization of equal load sharing given above, $(\sigma_I)_j = \sigma_I$ for all $j$. $(\sigma_B)_j$ is the tensile stress carried by the $j$th broken fiber at the plane of a matrix crack.

Were the contemplated specimen to be of finite size, the various quantities that appear in (6) would be random variables; since we have assumed an infinite number of fibers, these quantities have definite values for a given $\sigma_I$. This is because the probabilities computed below with the weakest link theory can be translated into fractions when the number of fibers is infinite. Computing $\sigma_{net}$ from eqn (6) requires the spatial distribution of fiber breaks, as well as the fiber stresses at the matrix–crack plane, given the position of the fiber breaks.

To explain how the various quantities in (6) are computed in terms of $\sigma_I$, it is useful to introduce the following stress distributions. First, we define $\sigma_0(z;\sigma_I)$ to be the stress at position $z$ in a fiber with no breaks; hence, by definition, $\sigma_0(0;\sigma_I) = \sigma_I$, when $z = 0$ coincides with a matrix–crack plane. The variation $\sigma_0(z;\sigma_I)$ reflects the load transfer to and from the fibers associated with the periodic matrix cracks; accordingly, $\sigma_0(z;\sigma_I)$ will have the same periodicity in $z$ as do the matrix cracks. For broken fibers, one can define a set of stress distributions in which $\sigma_k(z_1, z_2, \ldots, z_k, z; \sigma_I)$ is the stress at the point $z$ in a fiber that has breaks at the points $z_j$ $(j = 1, 2, \ldots, k)$. It is crucial to note that, in keeping with our generalization of the equal load sharing rule, the stress in a fiber depends only on the instant in the loading history $(\sigma_I)$ and on the position of *its* breaks, but not on the positions of its neighbor's breaks.

It is also important to point out that the actual functional forms of $\sigma_0(z;\sigma_I)$ and $\sigma_k(z_1, z_2, \ldots, z_k, z; \sigma_I)$ are *not* provided by this theory; they would be found from stress analyses involving fibers, matrix cracks and fiber breaks. This theory does, however, allow one to compute the ultimate strength, assuming the distributions $\sigma_0(z;\sigma_I)$ and $\sigma_k(z_1, z_2, \ldots, z_k, z)$ are known. This is quite valuable from a practical point of view, as very reasonable approximate stress distributions can be postulated based on one-dimensional analyses. Furthermore, as the results of more accurate analyses of fiber stresses become available, they can be incorporated into the theory via $\sigma_0(z;\sigma_f)$ and $\sigma_k(z_1,z_2,\ldots,z_k,z;\sigma_f)$.

In fact, in order to orient the reader, the specific forms for $\sigma_0(z;\sigma_f)$ and $\sigma_1(z_1,z;\sigma_f)$ which will be assumed for the numerical calculations are presented here, even though the equations from which the strength will be calculated will be given explicitly in terms of $\sigma_0(z;\sigma_f)$ and $\sigma_1(z_1,z;\sigma_f)$. The specific forms, which are based on the assumption of a constant interfacial shear stress, are given by:

$$\sigma_0(z;\sigma_f) = \sigma_f - \frac{2}{a}z\tau_{\text{int}} \qquad (0 < z < d) \tag{7a}$$

$$\sigma_1(z_1,z;\sigma_f) = \min\begin{cases}\sigma_0(z;\sigma_f) \\ \dfrac{2}{a}(z_1 - z)\tau_{\text{int}}\end{cases} \qquad (z < z_1) \tag{7b}$$

where $z$ is measured from the matrix–crack plane, the fiber break at $z_1$ is assumed to be in $z_1 > 0$, $2d$ is the matrix–crack spacing (see Fig. 3), $\tau_{\text{int}}$ is the interfacial shear stress, and $a$ is the fiber radius. Actually, $\sigma_0(z;\sigma_f)$ is the periodic extension of eqn (7a) which is consistent with the periodicity of the matrix cracks; thus, it corresponds to the familiar sawtooth distribution of stress. The form chosen for $\sigma_1(z_1,z;\sigma_f)$ reflects linear load transfer away from the break until the stress reaches $\sigma_0(z;\sigma_f)$, the level prevailing in the undisturbed portion of the composite. Thus, if it is needed, an analogous expression for $z > z_1$ may be derived.

As mentioned above, the distribution of fibers breaks is determined via the assumption of a weakest link law of fiber strength. Accordingly, it is assumed that there is a flaw function $n(\sigma)$ which is defined as follows: $n(\sigma)\,\mathrm{d}z$ is the probability that a segment of length $\mathrm{d}z$ has broken at or below the stress $\sigma$. Note that, in the common application of weakest link statistics to fibers, the fiber is treated as a one-dimensional continuum: instead of focusing on an elemental volume $\mathrm{d}V$, we focus on an elemental length $\mathrm{d}z$. We consider the particular case of the Weibull distribution in which $n(\sigma) = \alpha\sigma^m$. With the assumption of weakest link failure, determination of the necessary quantities in eqn (6) is a problem of combinatorics. An important ingredient in the theory is the distribution of fiber breaks; our method follows that of Oh and Finnie (1970) for predicting fracture locations. Details of the specific combinatorial arguments are presented in Schwietert and Steif (1990a); only abbreviated derivations are given here.

The contribution of the intact fibers to the load transmitted across the matrix–crack plane is given by:

$$\frac{1}{N_T}\sum_{i=1}^{N_I}(\sigma_I)_i = \sigma_f \exp\left[-\int_{-L}^{L} n(\sigma_0(z;\sigma_f))\,\mathrm{d}z\right] \tag{8}$$

where $2L$ is the composite specimen length. [Below, we explain why the integration limits differ from that in Schwietert and Steif (1990a).] This expression is arrived at by the usual argument that a length of fiber is intact if every one of its constituent segments is intact.

To compute the contribution of the broken fibers to the load transmitted across the matrix-crack plane, consider first the fibers that are broken once up to the stress $\sigma_f$; there are $N_{B_1}$ such fibers and their contribution is:

$$\frac{1}{N_T}\sum_{i=1}^{N_{B_1}}(\sigma_B)_i = \sigma_f\int_0^{\sigma_f}\int_{L}^{L}\exp\left[-\int_{-L}^{L} n(\sigma_0(z;\sigma_{f1}))\,\mathrm{d}z\right]\frac{\mathrm{d}n(\sigma_0(z_1;\sigma_{f1}))}{\mathrm{d}\sigma_{f1}}h(z_1)\,\mathrm{d}z_1\,\mathrm{d}\sigma_{f1}. \tag{9}$$

A portion of this multiple integral, namely

$$\exp\left[-\int_{-L}^{L} n(\sigma_0(z;\sigma_{f1}))\,dz\right]\frac{dn(\sigma_0(z_1;\sigma_{f1}))}{d\sigma_{f1}}\,dz_1\,d\sigma_{f1}$$

corresponds to the probability of a fiber being intact up to a stress $\sigma_{f1}$, and then failing in $z_1 < z < z_1+dz_1$ during the stress increment $d\sigma_{f1}$; this is identical to Oh and Finnie's equation (5). The function $h(z_1)$ [which is *unrelated* to Oh and Finnie's probability density $h(\sigma_m, \xi)$] is defined to be $\sigma_1(z_1, 0; \sigma_f)/\sigma_f$. Therefore, the product $h(z_1)\sigma_f$ has the following interpretation: it is the stress at the matrix-crack plane in a fiber which has a single break at $z_1$. From $\sigma_{f1}$ onwards, this broken fiber transmits a stress $h(z_1)\sigma_f$ across the plane $z = 0$. In general, $h(z_1)$ will depend on $\sigma_f$ even though $\sigma_f$ does not appear explicitly as an argument, as can be seen from the definition (7b). The contribution due to the once broken fibers, given by eqn (9), has a similar, though not identical, counterpart in Sutcu's (1989) theory; another notable difference in the theories is that his length, a sampling length, increases with the applied stress or with $\sigma_f$.

Similar combinatorial arguments can be used to find the contribution of the $N_{B_2}$ fibers that are broken twice up to the stress $\sigma_f$; the result is

$$\frac{1}{N_T}\sum_{i=1}^{N_{B_2}}(\sigma_B)_i = \sigma_f\int_0^{\sigma_f}\int_{-L}^{L}\exp\left[-\int_{-L}^{L} n(\sigma_0(z';\sigma_{f1}))\,dz'\right]\frac{dn(\sigma_0(z_1;\sigma_{f1}))}{d\sigma_{f1}}$$
$$\times\left\{\int_{\sigma_{f1}}^{\sigma_f}\int_{-z_1}^{z_1}\exp\left[-\int_{-z_1}^{z_1} R\{n(\sigma_1(z'',z_1;\sigma_{f2}))-n(\sigma_0(z'';\sigma_{f1}))\}\,dz''\right]\frac{dn(\sigma_1(z_2,z_1;\sigma_{f2}))}{d\sigma_{f2}}\right.$$
$$\left.\times H\{n(\sigma_1(z_2,z_1;\sigma_{f2}))-n(\sigma_0(z_2;\sigma_{f1}))\}(h(z_2)-h(z_1))\,dz_2\,d\sigma_{f2}\right\}dz_1\,d\sigma_{f1} \quad (10)$$

where

$$R(x) = \begin{cases} x & x > 0 \\ 0 & x \leqslant 0 \end{cases} \quad (11)$$

and

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0. \end{cases} \quad (12)$$

The second term of force balance (6), that is the contribution due to the broken fibers, is taken to be the sum of (9) and (10), meaning that up to two breaks in a fiber are permitted. More accurately, if the first break occurs at $z_1$, then we allow for the possibility of another break at $z_2$, where $-z_1 < z_2 < z_1$. Such fibers are loosely referred to as being "twice broken" fibers; strictly speaking, they may have broken more than twice, but these additional breaks are farther away from the matrix-crack plane than is the break at $z_2$. Similarly, a "once broken" fiber is one in which the very first break (which was at $z_1$) remains the break that is closest to the matrix plane. The effects of fiber breaks closer to the matrix crack plane than $z_2$ ("thrice broken" fibers, etc.) are not included in these calculations. In practice, it is generally unnecessary to consider more than one break in the length $L$; this was discovered through trial runs in which up to two breaks were included in the calculations.

Note that $h(z)$, which is connected with $\sigma_1(z_1, z; \sigma_f)$, appears in (10), while higher order distributions, in particular $\sigma_2(z_1, z_2, 0; \sigma_f)$, do not. This comes from the tacit, though unnecessary, assumption that $(\sigma_B)_i$ depends only on the break which is closest to the matrix-crack plane. This is tantamount to assuming that $\sigma_k(z_1, z_2, \ldots, z_k, 0; \sigma_f) = \sigma_1(z_{\min}, 0; \sigma_f)$, where $z_{\min}$ is equal to the $z_j$ $(j = 1, 2, \ldots, k)$ which has the minimum absolute value. Since the stress distributions are derived from simple shear lag analyses, there seems to be no obvious way of incorporating the effect of a farther break. If more refined stress distributions become available, however, then (10) could be modified to include this effect. In practice, the integrals in (8), (9) and (10) were evaluated numerically, using small increments in $\sigma_f$.

As defined earlier, $2L$ is the length of the specimen. There is another interpretation of $L$, which is based on its appearance as the limit of spatial integration in eqns (8), (9) and (10). Imagine the specimen to be longer than $2L$. In using the above equations to compute the stress acting across the matrix–crack plane at $z = 0$, we are, in effect, only accounting for fiber breaks which appear in the region $-L < z < L$. Due to the transfer of load back to the fiber with distance from a break, one expects breaks which are very far from the matrix–crack plane at $z = 0$ to have little influence. Consequently, increasing the parameter $L$ beyond a certain extent—which may still be much less than the actual specimen length—should have practically no effect on the predicted ultimate strength. In fact, this will be borne out by the results presented below; the predicted ultimate strength will be found to be independent of $L$, provided $L$ is sufficiently long.

The method presented here may be contrasted with our previous work (Schwietert and Steif, 1990a), where only breaks within one matrix–crack spacing were considered. That was clearly insufficient, and the error thereby incurred will be seen below. It is important to emphasize that this theory eliminates the necessity of invoking an ineffective length which features in Rosen's (1965) theory. Sutcu's (1989) theory tacitly begins with a force balance like (6), and results in expressions that are similar to (8) and (9). One essential difference is that his evaluation of the contributions due to intact and broken fibers requires choosing a sampling length, which is unnecessary here. Ideally, one wishes to have a theory which predicts the ultimate strength based only on the fiber strength statistics and on the interfacial shear strength. The present theory does precisely this; no length scales need to be explicitly introduced. The length which consistently appears in other theories, for example Sutcu's sampling length which is the maximum length of fiber that can be pulled out, is implicitly embedded in our eqn (7b). It is related to the distance from the fiber break at which our expression for the stress in a broken fiber changes from the linear load transfer proportional to $\tau_{\rm int}$ near the break to the undisturbed periodic stress distribution $\sigma_0(z; \sigma_f)$. Other than incorporating the piecewise nature of $\sigma_1(z_1, z; \sigma_f)$ properly into (9) and (10), however, we never need to invoke this length.

Finally, we introduce an additional element which is crucial to the theory. At the outset, the convention was adopted that "fiber stress" denotes the longitudinal stress averaged over the fiber cross-section. Since all the formulae involve only this averaged fiber stress, any deviations from uniformity would have been lost to the theory. Imagine that the stress over the fiber cross-section *were* very different from uniform: would one expect this to make any difference? The fiber stress enters the formulae for the net stress in two distinct ways: (i) as contributing to the load transmitted across the matrix–crack plane and (ii) through the flaw function $n(\sigma)$ which determines the probability of a break. Since $\sigma_{\rm net}$ is the stress averaged over the specimen cross-section, it surely can be determined from the load, or average stress, contributed by each fiber. On the other hand, following the suggestion of Sutcu (1989), we contemplate the possibility that the *distribution* of stress over the fiber cross-section, and not just the average, can seriously affect the probability of a break.

To see this, recall that fiber strength is generally thought to be controlled by surface flaws. Hence, the tensile stress in the outer part (near the surface) of the fiber would seem to be more critical to fiber strength than would the stress near the fiber core. This is clearly not an issue in the normal strength testing of fibers, in which case the stress is essentially uniform tension (at least in the gauge section). Why should one imagine that the stress in the fiber is ever much different from uniform? Because the fibers are spanning matrix *cracks*, and the matrix cracks impinge, in turn, upon the fibers. Hence, not only is the stress non-uniform, it might even be singular, at least within the theory of linear elasticity. One means of accounting for the impinging matrix cracks might be to do a fracture mechanics analysis involving the usual energy release rates.

Since the interfaces in composites which suffer multiple matrix cracking are weak, however, it would appear necessary to account for this weakness; surely the slippage at the interface has a blunting effect on the matrix cracks. In fact, the influence of a slipping interface on impinging cracks has been the subject of intensive study by the authors and co-workers (Dollar and Steif, 1989, 1991; Schwietert and Steif, 1989, 1990b). We have found that the stress concentrating power of such cracks depends sensitively on the character of the interface. Even though the precise magnitude of the stress enhancement is not yet known for the relevant crack configuration, it still seems appropriate to examine the possible effect of an enhanced stress near the fiber surface on the predicted ultimate strength of the composite. It is our belief that much of the complicated dependence of ultimate strength on the interface can be explained by appealing to the stress enhancement associated with the matrix cracks.

We account for the stress enhancement simply by modifying the stress distribution $\sigma_0(z;\sigma_f)$ as follows:

$$\sigma_0(z;\sigma_f) = \max\begin{cases} \sigma_f - \dfrac{2}{a} z\tau_{\text{int}} + \sigma_f\left(1 - \dfrac{z}{d_s}\right)c_s \\ \sigma_f - \dfrac{2}{a} z\tau_{\text{int}} \end{cases} \quad (0 < z < d). \tag{13}$$

Here, $c_s$ gives the stress concentration and $d_s$ gives the distance over which the stress is enhanced over the mean value. It should be understood that (13) applies only where the stress enters the flaw function $n(\sigma)$. Obviously, the stress enhancement near the surface comes at the expense of stress near the core, with the mean fiber stress being unaltered.

## RESULTS

In this section, we will compare the predictions of the ultimate tensile strength with experimental results for two composite systems: Nicalon†-reinforced lithium-alumino-silicate glass ceramic, SiC/LAS, which was tested by Prewo (1986) and reaction bonded silicon nitride reinforced by silicon carbide fibers (SCS-6‡), SiC/RBSN, which was tested by Bhatt (1989). In addition, we will examine the sensitivity of these predictions to material parameters.

Consider first SiC/LAS, in particular Prewo's sample no. 2369-7, for which the material data is presented in Table 1. Prewo determined the fiber strength and the strength distribution from fibers that were extracted from the composite material after fabrication. Using strength data based on extracted fibers seems sensible, as these data presumably reflect damage that occurs during fabrication. For this system, $\sigma_{\text{net}}$ versus $\sigma_f$ is presented in Fig. 4. This curve was generated by taking the interfacial shear stress $\tau_{\text{int}}$ to equal 3.0 MPa and letting the specimen half-length $L$ be $9d$, where the matrix-crack spacing $2d = 400$ $\mu$m (Marshall and Evans, 1985). It can be seen that the stress $\sigma_{\text{net}}$ increases with increasing $\sigma_f$ up to a maximum, and then starts to decrease. This maximum is the maximum stress that a specimen of this length can withstand.

Calculations of the maximum net stress were carried out for a range of specimen lengths. Figure 5 shows the maximum stress $\sigma_{\text{net}}$ as a function of the normalized specimen length $L/d$, for various values of the interfacial shear stress $\tau_{\text{int}}$. ($2d$ was held fixed at 400 $\mu$m.) Note that the maximum net stress decreases as the specimen length increases, and it approaches some fixed value as the specimen length becomes large. In the case of $\tau_{\text{int}} =$ 5 MPa, for example, the asymptotic limit is, for all practical purposes, reached when the specimen half-length is equal to five matrix-crack spacings. Note also that longer specimen lengths are required to approach the asymptotic limit when the interfacial shear stress is low; obviously, one needs to account for more distant breaks, which can still have an effect when the interfacial shear stress is low. This agrees with one's intuition that the length required for a broken fiber to regain its load is greater for low interfacial shear stresses. The greater this length, the greater will be the distance at which fiber breaks have a significant influence.

It is significant, and fortunate, that the values of $L$ at which the asymptotic limit is practically reached are far less than typical specimen lengths. The greater is $L$, the more

† Nippon Carbon Company.
‡ Textron Specialty Materials Division.

Table 1. Experimental data for SiC LAS, Prewo's sample no. 2369-7

| $V_f$ | $E_m/E_f$ | $a$ (μm) | $m$ | $\sigma_{UTS}$ (MPa) | $\sigma_f$ (MPa) | $L_f$ (mm) |
|---|---|---|---|---|---|---|
| 0.46 | 0.42 | 8.5 | 3.8 | 758 | 1580 | 25 |

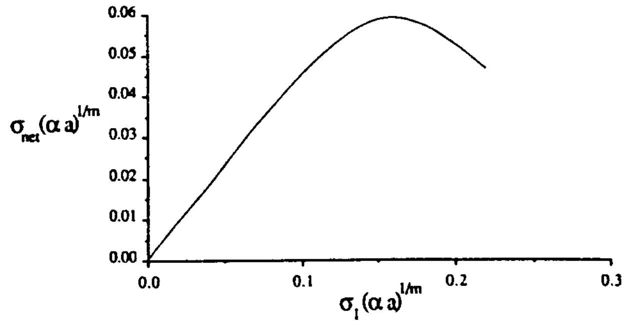


Fig. 4. Normalized composite net stress as a function of the stress in an intact fiber at the matrix-crack plane for SiC LAS ($L = 9d$).

time consuming are the numerical integrations necessary for computing $\sigma_{net}$ as a function of $\sigma_I$. Though one cannot precisely define it, the length at which the asymptotic limit is essentially reached ($L/d = 5$ for the case of $\tau_{int} = 5$ MPa) is significant in its own right. First, it gives one an idea of the minimum size composite which has essentially the same strength as the infinitely long, size-independent composite. Secondly, this length is certainly of the same order of magnitude as Rosen's ineffective length; that is, it is approximately equal to $a\sigma_f/(2\tau_{int})$. A more direct comparison between our prediction and the chain-of-bundles prediction is given below. It is crucial to appreciate, however, that there is no need to incorporate this length or any other length into our calculation; it falls out naturally.

It is also possible to keep track of the fraction of intact fibers, the fraction of "once broken" fibers, and the fraction of "twice broken" fibers. (That is, the fraction of fibers that are intact within the segment $-L < z < L$, and the fractions of fibers that are once broken and twice broken within that segment, ignoring breaks outside the segment.) These fractions are shown in Fig. 6 for the case of $L = 9d$ and an interfacial shear stress $\tau_{int} = 3.0$ MPa. The contribution of the twice broken fibers is found to be relatively small. This appears to be true for all the cases studied in this work, and that justifies neglecting higher order corrections to the contribution of the broken fibers. Unlike the prediction of the ultimate strength, which eventually becomes independent of $L$, the fractions of intact and broken fibers continue to change with increasing $L$. In fact, the fraction of intact fibers will continue to decrease with $L$ for a fixed level of $\sigma_I$. What will remain fixed is the fraction of intact fibers per unit length of composite.

Using the results from Fig. 5, the ultimate tensile strength (the asymptotic maximum net stress for increasing $L$) can be plotted as a function of the interfacial shear stress $\tau_{int}$

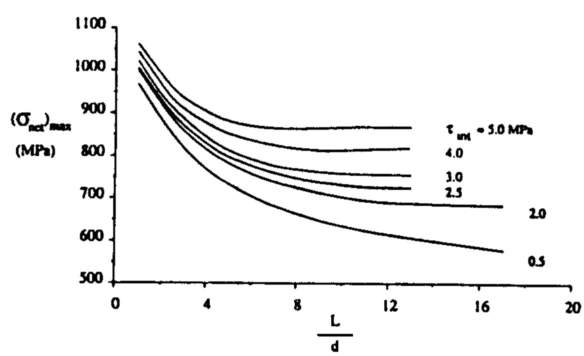


Fig. 5. Maximum net stress as a function of normalized specimen length for SiC/LAS.

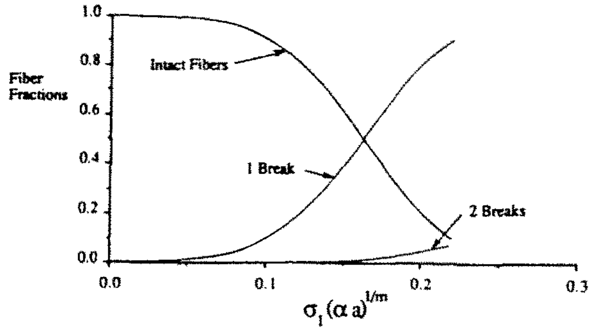


Fig. 6. Fraction of broken and intact fibers as a function of stress in an intact fiber at the matrix crack plane for SiC/LAS ($L = 9d$).

(Fig. 7). Consider now a comparison with Prewo's measured ultimate strength of 758 MPa. As can be seen from Fig. 7, the present theory would predict this ultimate tensile strength if the interfacial shear stress were approximately 3 MPa. This value compares reasonably well with the values of interfacial shear stress that have been measured in this system (Marshall and Evans, 1985).

Consider now the SiC/RBSN. Bhatt (1989) measured the tensile strength of this material (at a specimen-gauge length of 50 mm), and he repeated this measurement after the material had been submitted to a 100 h heat treatment at 600°C in oxygen. This heat treatment caused oxidation of the fiber–matrix interface, which decreased the apparent interfacial shear stress, as estimated from the matrix-crack spacing. In addition, the oxidation degraded the fiber surface coating and resulted in a decrease in fiber diameter and a decrease in fiber strength. A summary of the data given by Bhatt for the two cases (case 2a before the heat treatment and case 2b after the heat treatment) is presented in Table 2. Notice, however, that Bhatt determined the strength of the fibers in SiC/RBSN from the as-received SCS-6 fibers; to measure the strength of the fibers in the composite after the heat treatment, he performed a similar heat treatment on a batch of as-received fibers and determined their strength. In his interpretation, he assumed that the degradation in the average tensile strength of individual fibers that were heat treated was similar to that of the fibers in the composite.

The asymptotic limit of the maximum net stress, $\sigma_{UTS}$, that is predicted for these cases is 1036 and 631 MPa, respectively. Comparison of these values with the results of Table 2

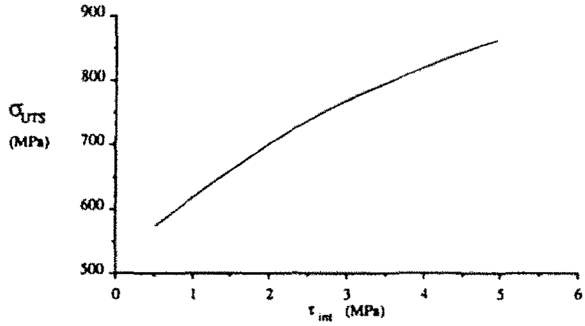


Fig. 7. Ultimate tensile strength as a function of interfacial shear stress $\tau_{int}$ for SiC/LAS.

Table 2. Experimental data for SiC/RBSN, before (a) and after (b) heat treatment

| Case | $V_f$ | $E_m/E_f$ | $a$ (μm) | $d$ (μm) | $m$† | $\tau$ (MPa) | $\sigma_{UTS}$ (MPa) | $\sigma_f$ (MPa) | $L_f$ (mm) |
|---|---|---|---|---|---|---|---|---|---|
| a | 0.3 | 0.275 | 71 | 400 | 8 | 18 | 682 | 3800 | 50 |
| b | 0.3 | 0.275 | 71 | 6000 | 8 | 0.8 | 270 | 3200 | 50 |

† Bhatt does not provide a measure for $m$ with his fiber strength measurements in the cited reference. This value for $m$ was provided through personal communication.

shows that for both cases the predictions considerably overestimate the actual strength. In an attempt to explain this discrepancy we first examined the sensitivity of the model predictions to several of the assumptions made in developing the theory. In particular, we considered the assumption of using a constant shear stress model for $\sigma_0(z;\sigma_f)$, the assumption of equal matrix–crack spacing and the assumption that the load transmitted across the matrix–crack plane by the broken fibers is only dependent on the break closest to the matrix–crack plane. We found that these assumptions had very little effect on our predictions. [More details can be found in Schwietert (1990).] It is also possible that inaccuracies in the constituent parameters could explain the differences between the predicted and the measured values. To investigate this, we assess the sensitivity of the predictions to variations in the Weibull modulus $m$, the interfacial shear stress $\tau_{int}$, and the mean fiber strength $\sigma_f$. Furthermore, the influence of a possible stress concentration in the fibers near the matrix–crack planes was examined.

First, the Weibull modulus $m$ was varied, while holding the mean strength at the original gauge length fixed; this requires $\alpha$ to be altered with $m$. Figure 8 shows the predicted ultimate tensile strength of SiC/LAS, normalized by the mean fiber strength (presented in Table 1), as a function of the Weibull parameter $m$ ($\tau_{int}$ fixed at 3.0 MPa). It can be seen from Fig. 8 that the composite strength diminishes slightly when the fibers have a more consistent strength. The analogous curves for SiC/RBSN are presented in Fig. 8 as well. With increasing $m$, the strength of SiC/RBSN case 2b increases, while the strength of SiC/RBSN case 2a first decreases, and then increases. An explanation of this effect is given in Schwietert and Steif (1990).

We considered next the influence of the interfacial shear stress on the predictions. The results of the prediction for SiC/LAS with changing $\tau_{int}$ are presented in Fig. 7; these results show that the predicted strength increases with increasing interfacial shear stress. This trend can be understood if one considers the influence of the interfacial shear stress on the stress distributions (7). The fiber stress, as given by $\sigma_0(z;\sigma_f)$, varies more rapidly with increasing interfacial shear stress. Clearly, as $\tau_{int}$ increases, the average axial fiber stress decreases in some regions, and increases nowhere. Furthermore, the interfacial shear stress controls the degree to which a fiber break diminishes the load-carrying capacity of the composite, through its influence on the distribution of the stress in a broken fiber, $\sigma_1(z_1,z;\sigma_f)$. In particular, when the interfacial shear stress is higher, the fiber regains its stress more quickly from the break. Hence, the ultimate strength increases with increasing $\tau_{int}$. Similar calculations were performed for the SiC/RBSN system, and similar trends were found. However, the results also showed that inaccuracies in the interfacial shear stress can only explain a small part of the differences between the predictions and the measured values.

Next, the dependence of the predictions on the mean fiber strength $\sigma_f$ was examined. Experiments by Prewo (1986) on Nicalon fibers demonstrate that the fibers do degrade significantly during the fabrication of the composite. Prewo (1986) tested these fibers before and after fabrication into SiC/LAS; for a gauge length of 25 mm, he measured mean fiber strengths of 2300 and 1580 MPa, respectively. Since the SCS-6 fibers in the SiC/RBSN specimen were tested before the fabrication, it seems likely that the actual strength of the fibers inside the composite is less than the values presented in Table 2.

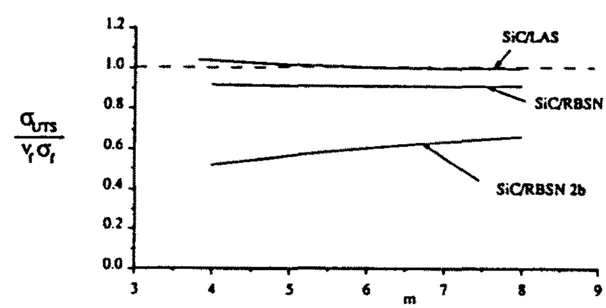


Fig. 8. Normalized ultimate tensile strength as a function of the Weibull modulus.

To examine the effect of having an incorrect value for fiber strengths, we repeated the calculations for the SiC/RBSN system, assuming different values for $\sigma_f$. As expected, the results indicate that the predicted ultimate tensile strength is nearly proportional to the mean fiber strength. In fact, the predictions would agree with the measured ultimate tensile strengths, if $\sigma_f$ were taken to be 2270 MPa for SiC/RBSN case 2a and 1200 MPa for SiC/RBSN case 2b. Clearly, the accuracy of this parameter has an important influence on the predictions of the theory set forth here, and this can explain some of the differences between theory and experiment.

Finally, we examined the influence of a stress concentration in the fibers. The calculations of the strength of SiC/RBSN case 2a were repeated with the stress distribution (13) replacing (7a). The results of these calculations are presented in Table 3, which shows the ultimate tensile strength predictions for different values of $c_s$ and $d_s$. Previous studies of cracks impinging upon weak interfaces (Dollar and Steif, 1989, 1991; Schwietert and Steif, 1989, 1990b) suggest that the values for $c_s$ and $d_s$ are somewhere in the range presented in Table 3. (As discussed earlier, $c_s$ is a measure of the enhancement of the stress at the fiber surface; $d_s$ is the length over which it is enhanced.) A stress concentration can influence the strength in two ways. First, since the stress at the fiber surface increases over some regions, and decreases nowhere, the number of breaks for a given $\sigma_f$ increases. Secondly, since the increase in stress occurs near the matrix–crack plane, there will be more fiber breaks in the vicinity of the central matrix–crack plane (and near other matrix–crack planes); these broken fibers will contribute less to the load transmitted across the central matrix–crack plane. Both effects tend to decrease the composite strength, and the results of Table 3 indicate that, for this system, even a small stress concentration over a relatively short distance can bring the predicted values down considerably. By contrast, the sensitivity to a stress concentration is lower in the Nicalon-based system, where $m$ is lower ($m = 3.8$). Greater variability in the fiber strength makes the composite less susceptible to locally enhanced stresses.

The studies of cracks impinging upon weak interfaces (Dollar and Steif, 1989, 1991; Schwietert and Steif, 1989, 1990b) also suggest that the stress concentration in the fibers increases with increasing interfacial shear stress. This means that the stress concentration parameters $c_s$ and $d_s$ are related to $\tau_{int}$. Therefore, we repeated the calculations for SiC/RBSN case 2a, and increased the interfacial shear stress and the stress concentration ($c_s$, $d_s$) in the fiber simultaneously. Since there is insufficient quantitative information available to determine precisely how $c_s$ and $d_s$ would change with increasing $\tau_{int}$, this necessarily involves some guesswork. The mean fiber strength was chosen to be 2270 MPa (at a gauge length of 50 mm). The results of these calculations are presented in Fig. 9. These results demonstrate that the prediction of the ultimate tensile strength first increases with increasing interfacial shear stress, and then diminishes. Clearly, the effect of the stress concentration becomes more important with increasing $\tau_{int}$, possibly reducing the composite strength.

Although the variation of $c_s$ and $d_s$ with $\tau_{int}$ is speculative, the results of Fig. 9 are important. They represent a possible explanation for the complex dependence on the interfacial shear strength that has been observed experimentally. We suggest that there are two fundamental means by which the interfacial shear stress affects the composite strength. First, the interfacial shear stress sets the load transfer rate. When the interfacial shear stress

Table 3. *Predictions for the ultimate tensile strength of SiC/RBSN case 2a (in MPa), including a stress concentration ($c_s$, $d_s$) in the fiber*

| | $d_s/a$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $c_s$ | 0.5 | 1.0 | 1.5 | 2.0 | 3.0 | 4.0 | 7.0 |
| 0.1 | 1032 | 1028 | 1026 | 1023 | 1017 | 1012 | 999 |
| 0.25 | 1024 | 1012 | 1002 | 994 | 978 | 963 | 938 |
| 0.5 | 992 | 963 | 941 | 924 | 898 | 878 | 839 |
| 1.0 | 872 | 822 | 789 | 766 | 737 | 716 | 679 |
| 2.0 | 640 | 600 | 574 | 556 | 532 | 515 | 488 |
| 3.0 | 427 | 395 | 373 | 358 | 340 | 328 | 309 |

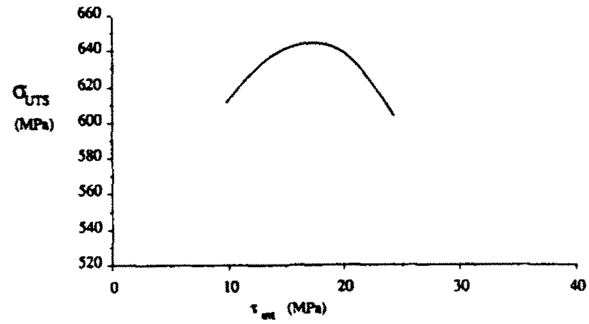


Fig. 9. Ultimate tensile strength as a function of the interfacial shear stress, including the effect of increasing stress concentration.

is increased, a broken fiber regains its load faster; this tends to increase the composite strength. [Sutcu's (1989) model and Rosen's (1965) chain-of-bundles model capture this effect at least qualitatively.] The second major influence of the interface is to control the stress concentrating effect of matrix cracks. When the interfacial shear stress is increased, the stress near the fiber surface is increased; this tends to decrease the composite strength. This second effect may be more familiar in a slightly different context. It is generally believed that a relative low interfacial shear stress is beneficial in brittle-matrix composites, because it leads to deflection of matrix cracks at fibers. As suggested recently by Dollar and Steif (1991), it may be more useful to compare different interfaces on the basis of their differing tendencies to cause a stress concentration. In applying this idea to ultimate strength, we are distinguishing between different interfacial shear strengths, all of which are sufficiently low to allow multiple matrix cracking.

Returning to Fig. 9, one can see that the load transfer effect dominates at low interfacial shear strengths, whereas the stress concentration effect dominates at higher interfacial shear strengths. Consider, for example, an SiC/LAS which has been subjected to an oxidizing environment; this removes the carbon surface layer and bonds the fiber to the matrix. With the typical degree of oxidation, this tends to reduce the composite strength. The stress concentration effect seems to be dominating here, even to the extent that multiple matrix cracking may not be permitted. On the other hand, oxidizing SiC/RBSN, which seems to remove the carbon layer leaving the smaller diameter fiber to rattle in its matrix socket, results in a loss of composite strength. In this case, the load transfer effect appears to dominate (though loss in mean fiber strength is always a possibility which must be considered). The results of Lowden (1990) on Nicalon-reinforced silicon carbide appear to be quite consistent with our proposed explanation of interfacial shear strength dependence. At low interfacial shear strengths, the composite strength increases with the interfacial shear strength, reflecting the dominance of the load transfer mechanism. At high interfacial shear strengths, composite strength decreases with the interfacial shear strength, reflecting the dominance of the stress concentration mechanism. Clearly, this suggests that there is, in fact, an optimum interfacial shear strength, though a considerable amount of work remains to be done to identify that optimum.

Finally, the predictions of the present theory are compared with Rosen's readily used chain-of-bundles model. Obviously, this comparison is only fair if effects due to stress concentrations are neglected. (In composite systems where the stress concentration effects are significant, the chain-of-bundles model would clearly be inadequate.) In this comparison, we assume that the interfacial shear stress, $\tau_{int}$, and the fiber strength distribution are given. Rosen's chain-of-bundles model requires a calculation of the ineffective length, $L_{ineff}$, which is the length over which some percentage (e.g. 90%) of the load is transferred back to the fiber. For a Weibull distribution, Rosen's prediction of the composite strength is then given by (5).

In Rosen's original model, the ineffective length is inferred from a shear-lag analysis of a fiber which is elastically bonded to a matrix (which is why a percentage less than 100% must be used in defining the ineffective length). For a composite with a fixed interfacial shear stress $\tau_{int}$, the ineffective length is generally defined to be

$$L_{\text{ineff}} = \frac{a\sigma_f}{2\tau_{\text{int}}} \tag{14}$$

where $\sigma_f$ is the mean fiber strength. This is the length necessary to transfer a load which is equal to $\pi a^2\sigma_f$. For a Weibull distribution, the mean fiber strength actually depends on length; we, therefore, take $\sigma_f$ to be the mean fiber strength for a fiber of length $L_{\text{ineff}}$, which implies

$$L_{\text{ineff}} = \left(\frac{\Gamma(1+1/m)a}{2\tau_{\text{int}}(\alpha)^{1/m}}\right)^{m/m+1}. \tag{15}$$

The chain-of-bundles prediction would emerge from substituting (15) into (5).

The chain-of-bundles prediction is compared with our predicted $\sigma_{UTS}$ for the same set of parameters $\alpha$, $m$ and $\tau_{\text{int}}$ by asking the inverse question: what fiber bundle has the same strength as our predicted $\sigma_{UTS}$? The fibers in this bundle have length $L_{UTS}$, where

$$L_{UTS} = \frac{1}{\alpha m e\left(\dfrac{\sigma_{UTS}}{V_f}\right)^m}. \tag{16}$$

If $L_{UTS}$ were equal to $L_{\text{ineff}}$, then our theory and the chain of bundles model would agree. To compare these lengths, the quantity $\beta$ is formed:

$$\beta = \frac{2L_{UTS}\tau_{\text{int}}}{a\sigma_f} \tag{17}$$

where $\sigma_f$ is the mean strength of a fiber of length $L_{UTS}$. If $\beta$ were found to equal 1, then the lengths would be the same; instead, we found values of $\beta$ to be between 0.36 and 0.59. The difference between $\beta$ and 1 seems to be the error that Rosen incurred in reconfiguring the distribution of breaks into a chain of bundles. In fact, we can use Rosen's chain of bundles prediction (5) if we take the ineffective length equal to $L_{\text{bun}}$, which is defined as

$$L_{\text{bun}} = \left(\frac{\Gamma(1+1/m)\beta a}{2\tau_{\text{int}}(\alpha)^{1/m}}\right)^{m/m+1} \tag{18}$$

instead of (15). Choosing $\beta$ to be equal to 0.50 consistently gives results which are within 5% of our predictions, provided any stress concentration effects can be safely neglected.

## CONCLUSIONS

A theory for ultimate tensile strength which is particularly suited to ceramic–matrix composites exhibiting multiple matrix cracking has been presented. This theory accounts for the random failure of fibers at flaws, and it utilizes a generalization of the notion of equal load sharing. The theory was used to predict the ultimate tensile strength for two composite systems. In the case of a Nicalon-reinforced lithium alumino-silicate glass–matrix composite, the theory agrees well with experiment, provided the *in situ* fiber strength is used (for instance, as measured on fibers extracted after composite fabrication). By contrast, the agreement with some experimental results for SiC-reinforced, silicon nitride composite is quite poor. To understand the possible sources of discrepancy, the sensitivity of the predictions to key material parameters, including fiber strength, fiber strength variability, and interfacial shear strength, was investigated. In addition, the effect of a locally enhanced stress at the fiber surface near the matrix–crack planes was considered. The two most likely sources of the discrepancy appear to be neglecting the loss in fiber strength associated with fabrication, and neglecting the stress enhancement associated with the matrix cracks. It is proposed that the stress concentration associated with matrix cracks could explain the non-monotonic dependence of composite strength on interfacial shear stress which is observed in some systems.

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